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Cohomology structure for a Poisson algebra: II

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Abstract For a Poisson algebra, we prove that the Poisson cohomology theory introduced by Flato et al. (1995) is given by a certain derived functor. We show that the (generalized) deformation quantization is equivalent to the formal deformation for Poisson algebras under certain mild conditions. Finally we construct a long exact sequence, and use it to calculate the Poisson cohomology groups via the Yoneda-extension groups of certain quasi-Poisson modules and the Lie algebra cohomology groups.

Keywords Poisson algebra, Poisson cohomology, formal deformation, deformation quantization

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1 Introduction

Flato et al. [5] developed a cohomology theory and a formal deformation theory for Poisson algebras (not necessarily commutative). They showed that this cohomology controls those formal deformations such that the associative multiplication and the Lie bracket are simultaneously deformed. We call this the cohomology Flato-Gerstenhaber-Voronov (FGV)-Poisson cohomology, or simply Poisson cohomology.

An immediate question is whether the FGV-Poisson cohomology is exactly given by usual Yoneda-extensions or derived functors. We introduce a complex $\chi_\bullet(A)$ of free Poisson modules for a Poisson algebra A, and show that the FGV-Poisson cohomology is exactly defined by the derived functor associated with the complex, which gives an affirmative answer to the above question.

Theorem 1.1 (See Theorem 3.5). *Let A be a Poisson algebra and M be a Poisson module over A. Then*

$$\mathrm{HP}^n(A, M) \cong \mathrm{H}^n\mathrm{Hom}_{\mathcal{P}}(\chi_\bullet(A), M).$$

An interesting observation is that Kontsevich's deformation quantization can be interpreted as a certain formal deformation of Poisson algebras. As a consequence, we may give a necessary condition for the existence of the deformation quantization by using the FGV-Poisson cohomology groups. To be precise, a Poisson algebra has no quantization deformations whenever the second FGV-Poisson cohomology group vanishes.

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Proposition 1.2 (See Proposition 4.6)**.** *Let $P = (A, \cdot, \{-,-\})$ be a nontrivial commutative Poisson algebra. If $\mathrm{HP}^2(A) = 0$, then P has no deformation quantization.*

A well-known result by Farkas and Letzter says that a prime noncommutative algebra admits only standard Poisson structures (see [4, Theorem 1.2]). In some sense, this result will lead to a problem of lack of (non-standard) noncommutative Poisson algebras, and for this reason more general Poisson structures have been introduced and studied (see [15,17]). While in the study of the deformation quantization, it will have an unexpected application. It guarantees the existence of the (generalized) deformation quantization under the certain condition.

Theorem 1.3 (See Theorem 4.4)**.** *Let $P = (A, \cdot, \{-,-\})$ be a commutative Poisson algebra.*

(1) *If P admits an n-deformation quantization for some $n \geqslant 1$, then P can be deformed to a standard Poisson algebra.*

(2) *Assume further that each formal deformation $(A[[t]], m_t)$ of $(A, \cdot)$ has only standard Poisson structures. Then P has an n-deformation quantization for some positive integer n if and only if P has a formal deformation.*

In particular, as it is shown in Example 4.5, if a Poisson algebra is an integral domain as an associative algebra, then it has a (generalized) deformation quantization if and only if it has nontrivial deformation.

Recall that for a commutative Poisson algebra, Lichnerowicz [11] has introduced a cohomology theory, which we call the Lichnerowicz-Poisson cohomology (LP-cohomology for short). We would like to mention that for commutative Poisson algebras, the FGV-Poisson cohomology is different from the LP-cohomology. Roughly speaking, the FGV-Poisson cohomology controls the formal deformations which deform the product and the Lie bracket simultaneously, while the LP-cohomology controls those ones which only deform the Lie bracket. In Section 5, we compare the FGV-Poisson cohomology groups and the LP-cohomology groups in lower degrees for commutative Poisson algebras.

The FGV-Poisson cohomology groups are quite nontrivial to calculate in general. However, we may construct a long exact sequence, involving the FGV-Poisson cohomology groups, the Yoneda-extension groups of certain quasi-Poisson modules and the Lie algebra cohomology groups (see Theorem 6.4). This enables us to calculate the FGV-cohomology via the Lie algebra cohomology and the quasi-Poisson cohomology, where the former has been well studied by many authors, and the latter was discussed in our previous paper [1].

The rest of the paper is organized as follows. In Section 2, we recall some basics. In Section 3 we construct a bicomplex of free Poisson modules for a Poisson algebra, whose total complex applies to compute the FGV-Poisson cohomology groups. Section 4 explains how Kontsevich's deformation quantization is related to the formal deformation of Poisson algebras, and some conditions of the existence of the (generalized) deformation quantization are also provided there in the language of the FGV-Poisson cohomology. In Section 5, we compare the FGV-Poisson cohomology with the LP-cohomology for commutative Poisson algebras. In Section 6, we show a long exact sequence, and apply it to calculate the FGV-Poisson cohomology groups by using the Lie algebra cohomology and the quasi-Poisson cohomology. In the last section, we study the standard Poisson algebra of the 2×2 matrix algebra. We compute its Poisson cohomology groups of lower degrees. Moreover, we show that in this case, any Poisson 2-cocycle lifts to a formal deformation.

Throughout, $\Bbbk$ will be a field of characteristic zero, all the associative algebras over $\Bbbk$ have a multiplicative identity element, and all the unadorned Hom and $\otimes$ will be $\mathrm{Hom}_{\Bbbk}$ and $\otimes_{\Bbbk}$, respectively. For simplicity, we denote by A^i and $\wedge^j$ the tensor product and the j-th exterior power of the $\Bbbk$-space A, respectively.

2 Preliminaries

A triple $(A, \cdot, \{-,-\})$ is called a *Poisson algebra* over $\Bbbk$, if $(A, \cdot)$ is an associative $\Bbbk$-algebra (not necessarily commutative), $(A, \{-,-\})$ is a Lie algebra over $\Bbbk$, and the Leibniz rule $\{ab, c\} = a\{b, c\} + \{a, c\}b$ holds for all $a, b, c \in A$. A *quasi-Poisson A-module* M is both an A-A-bimodule and a Lie module over $(A, \{-,-\})$

with the action given by $\{-,-\}_*: A \times M \to M$, which satisfies

$$\{a, bm\}_* = \{a, b\}m + b\{a, m\}_*,$$
$$\{a, mb\}_* = m\{a, b\} + \{a, m\}_* b$$

for all $a, b \in A$ and $m \in M$. In addition, if

$$\{ab, m\}_* = a\{b, m\}_* + \{a, m\}_* b$$

holds for all $a, b \in A$ and $m \in M$, then M is called a *Poisson A-module*. Let M and N be (quasi-)Poisson modules. A homomorphism of (quasi-)Poisson A-modules is a $\Bbbk$-linear function $f: M \to N$ which is a homomorphism of both A-A-bimodules and Lie modules.

The following convention is handy in calculation, and we refer to [18] for more details.

Denote by A^{op} the opposite algebra of the associative algebra A. To avoid confusion, we usually use a to denote an element in A and a' its corresponding element in A^{op}. Denote by A^e the *enveloping algebra* $A \otimes A^{\mathrm{op}}$ of the associative algebra $(A, \cdot)$, and by $\mathcal{U}(A)$ the *universal enveloping algebra* of the Lie algebra $(A, \{-,-\})$. It is well known that the category of A-A-bimodules is isomorphic to the A^e-module category and the category of Lie modules over $(A, \{-,-\})$ is isomorphic to the $\mathcal{U}(A)$-module category. Note that $\mathcal{U}(A)$ is a cocommutative Hopf algebra with the comultiplication, the counit map, and the antipode defined by

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = 0 \quad \text{and} \quad S(a) = -a$$

for all $a \in A$. Then A^e is a $\mathcal{U}(A)$-module algebra with the action given by

$$\alpha(a \otimes b') = \sum \alpha_1(a) \otimes (\alpha_2(b))'$$

for $\alpha \in \mathcal{U}(A)$, $a \otimes b' \in A^e$, where

$$\alpha(a) = \{x_1, \{x_2, \dots, \{x_n, a\} \cdots\}\}$$

for $\alpha = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \mathcal{U}(A)$ and $a \in A$.

Definition 2.1 (See [18]). Let A be a Poisson algebra. The smash product $A^e \# \mathcal{U}(A)$ is called the *quasi-Poisson enveloping algebra* of A and denoted by $\mathcal{Q}$. The *Poisson enveloping algebra* of A, denoted by $\mathcal{P}$, is defined as the quotient algebra $\mathcal{Q}/J$, where J is the ideal of $\mathcal{Q}$ generated by

$$1_A \otimes 1_A' \#(ab) - a \otimes 1_A' \# b - 1_A \otimes b' \# a$$

for all $a, b \in A$.

Theorem 2.2 (See [18]). *The category of quasi-Poisson modules over A is isomorphic to the category of left $\mathcal{Q}$-modules, and the category of Poisson modules over A is isomorphic to the category of left $\mathcal{P}$-modules.*

Given a quasi-Poisson A-module M, one can define a $\mathcal{Q}$-module M by setting $(a \otimes b' \# \alpha)m = a\alpha(m)b$ for all $m \in M$ and $a \otimes b' \# \alpha \in \mathcal{Q}$. Conversely, given a left $\mathcal{Q}$-module M, we set

$$am = (a \otimes 1_A' \# \mathbf{1})m, \quad ma = (1_A \otimes a' \# \mathbf{1})m \quad \text{and} \quad \{a, m\}_* = (1_A \otimes 1_A' \# a)m$$

for all $m \in M, a \in A$ and give a quasi-Poisson A-module structure on M, where $\mathbf{1}$ is the multiplicative identity element of $\mathcal{U}(A)$. The correspondence of Poisson modules and $\mathcal{P}$-modules is given similarly. For simplicity, we write $a \otimes b' \# \alpha + J$ as $a \otimes b' \# \alpha$ as in $\mathcal{P}$ when no confusion can arise.

In [1], Bao and Ye considered the following bicomplex by taking the tensor product of the bar resolution of the A^e-module A and the projective resolution of the trivial $\mathcal{U}(A)$-module $\Bbbk$:

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & A^4\otimes\mathcal{U}(A) & \xleftarrow{\eta^H_{2,0}} & A^4\otimes\mathcal{U}(A)\otimes\wedge^1 & \xleftarrow{\eta^H_{2,1}} & A^4\otimes\mathcal{U}(A)\otimes\wedge^2 & \longleftarrow & \cdots \\
 & & {\scriptstyle\eta^V_{1,0}}\downarrow & & {\scriptstyle\eta^V_{1,1}}\downarrow & & {\scriptstyle\eta^V_{1,2}}\downarrow & & \\
0 & \longleftarrow & A^3\otimes\mathcal{U}(A) & \xleftarrow{\eta^H_{1,0}} & A^3\otimes\mathcal{U}(A)\otimes\wedge^1 & \xleftarrow{\eta^H_{1,1}} & A^3\otimes\mathcal{U}(A)\otimes\wedge^2 & \longleftarrow & \cdots . \\
 & & {\scriptstyle\eta^V_{0,0}}\downarrow & & {\scriptstyle\eta^V_{0,1}}\downarrow & & {\scriptstyle\eta^V_{0,2}}\downarrow & & \\
0 & \longleftarrow & A^2\otimes\mathcal{U}(A) & \xleftarrow{\eta^H_{0,0}} & A^2\otimes\mathcal{U}(A)\otimes\wedge^1 & \xleftarrow{\eta^H_{0,1}} & A^2\otimes\mathcal{U}(A)\otimes\wedge^2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.1}
$$

Its total complex gives a free resolution of A as a $\mathcal{Q}$-module. For any quasi-Poisson module M, after applying the functor $\mathrm{Hom}_{\mathcal{Q}}(-,M)$ to the total complex, we obtain the quasi-Poisson complex with coefficients in M, whose cohomology group is isomorphic to the extension group $\mathrm{Ext}^*_{\mathcal{Q}}(A,M)$, and is called the *quasi-Poisson cohomology group with coefficients in* M (see [1, Section 3]).

3 The FGV-Poisson cohomology

3.1 A complex of Poisson modules

Let $(A,\cdot,\{-,-\})$ be a Poisson algebra and $\mathcal{P}$ be the Poisson enveloping algebra of A. Set

$$
C_{i,j}=\begin{cases}\mathcal{P}\otimes\wedge^j, & i=0,\quad j\geqslant 0,\\ \mathcal{P}\otimes A^{i+1}\otimes\wedge^{j-1}, & i\geqslant 1,\quad j\geqslant 1,\\ 0, & \text{otherwise.}\end{cases}
$$

Clearly, $C_{i,j}$ is a left $\mathcal{P}$-module for any $i,j\geqslant 0$. We consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathcal{P}\otimes A^3 & \xleftarrow{\delta^H_{2,1}} & \mathcal{P}\otimes A^3\otimes\wedge^1 & \xleftarrow{\delta^H_{2,2}} & \mathcal{P}\otimes A^3\otimes\wedge^2 & \longleftarrow & \cdots \\
 & & {\scriptstyle\delta^V_{1,1}}\downarrow & & {\scriptstyle\delta^V_{1,2}}\downarrow & & \downarrow{\scriptstyle\delta^V_{1,3}} & & \\
0 & \longleftarrow & \mathcal{P}\otimes A^2 & \xleftarrow{\delta^H_{1,1}} & \mathcal{P}\otimes A^2\otimes\wedge^1 & \xleftarrow{\delta^H_{1,2}} & \mathcal{P}\otimes A^2\otimes\wedge^2 & \longleftarrow & \cdots , \\
 & & {\scriptstyle\delta^V_{0,1}}\downarrow & & {\scriptstyle\delta^V_{0,2}}\downarrow & & \downarrow{\scriptstyle\delta^V_{0,3}} & & \\
\mathcal{P} & \xleftarrow{\delta^H_{0,0}} & \mathcal{P}\otimes\wedge^1 & \xleftarrow{\delta^H_{0,1}} & \mathcal{P}\otimes\wedge^2 & \xleftarrow{\delta^H_{0,2}} & \mathcal{P}\otimes\wedge^3 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\delta^V_{i,j}: C_{i+1,j}\to C_{i,j}$ and $\delta^H_{i,j}: C_{i,j+1}\to C_{i,j}$ are $\mathcal{P}$-homomorphisms given by

$$
\begin{aligned}
&\delta^V_{i,j}((1_A\otimes 1'_A\#\mathbf{1})\otimes a_0\otimes\cdots\otimes a_{i+1}\otimes\omega^{j-1})\\
&\quad=(-1)^i\bigg((a_0\otimes 1'_A\#\mathbf{1})\otimes a_1\otimes\cdots\otimes a_{i+1}\otimes\omega^{j-1}
\end{aligned}
$$

$$+\sum_{k=0}^{i}(-1)^{k+1}(1_A\otimes 1'_A\#\mathbf{1})\otimes a_0\otimes\cdots\otimes a_ka_{k+1}\otimes\cdots\otimes a_{i+1}\otimes\omega^{j-1}$$
$$+(-1)^i(1\otimes a'_{i+1}\#\mathbf{1})\otimes a_0\otimes\cdots\otimes a_i\otimes\omega^{j-1}\Big)$$

for $i\geqslant 1$, $j\geqslant 0$,

$$\begin{aligned}\delta^V_{0,j}&((1_A\otimes 1'_A\#\mathbf{1})\otimes a_0\otimes a_1\otimes\omega^{j-1})\\ &=(a_0\otimes 1'_A\#\mathbf{1})\otimes(a_1\wedge\omega^{j-1})-(1_A\otimes 1'_A\#\mathbf{1})\otimes(a_0a_1\wedge\omega^{j-1})\\ &\quad+(1_A\otimes a'_1\#\mathbf{1})\otimes(a_0\wedge\omega^{j-1})\end{aligned}$$

for $j\geqslant 1$, $\omega^{j-1}\in\wedge^{j-1}$, and

$$\begin{aligned}\delta^H_{i,j}&((1_A\otimes 1'_A\#\mathbf{1})\otimes\theta^{i+1}\otimes(x_0\wedge\cdots\wedge x_{j-1}))\\ &=\sum_{k=0}^{j-1}(-1)^k(1_A\otimes 1'_A\#x_k)\otimes\theta^{i+1}\otimes(x_0\wedge\cdots\widehat{x_k}\cdots\wedge x_{j-1})\\ &\quad+\sum_{0\leqslant p<q\leqslant j-1}(1_A\otimes 1'_A\#\mathbf{1})\otimes\theta^{i+1}\otimes(\{x_p,x_q\}\wedge x_0\wedge\cdots\widehat{x_p}\cdots\widehat{x_q}\cdots\wedge x_{j-1})\end{aligned}$$

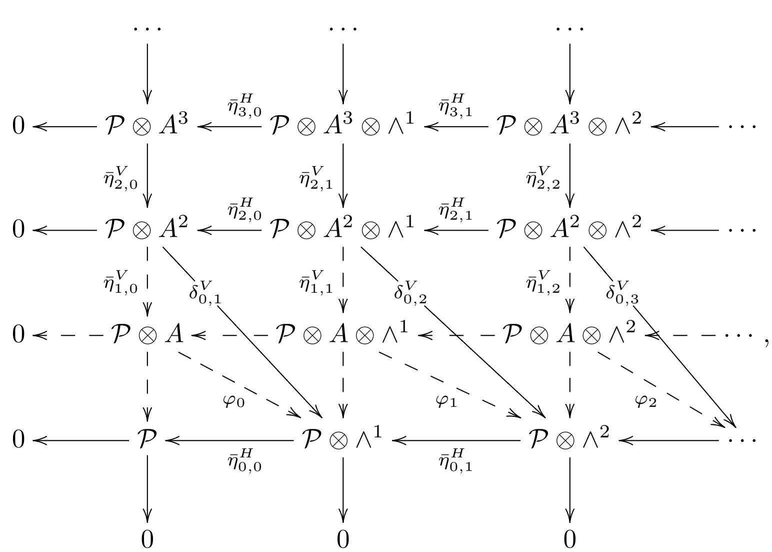
for any $i\geqslant 1$, $j\geqslant 1$, $\theta^{i+1}\in A^{i+1}$,

$$\begin{aligned}\delta^H_{0,j}&((1_A\otimes 1'_A\#\mathbf{1})\otimes(x_0\wedge\cdots\wedge x_j))\\ &=\sum_{k=0}^{j}(-1)^k(1_A\otimes 1'_A\#x_k)\otimes(x_0\wedge\cdots\widehat{x_k}\cdots\wedge x_j)\\ &\quad+\sum_{0\leqslant p<q\leqslant j}(1_A\otimes 1'_A\#\mathbf{1})\otimes(\{x_p,x_q\}\wedge x_0\wedge\cdots\widehat{x_p}\cdots\widehat{x_q}\cdots\wedge x_j)\end{aligned}$$

for any $j\geqslant 0$.

Proposition 3.1. *The diagram* $C_{\bullet,\bullet}=(C_{i,j},\delta^H_{i,j},\delta^V_{i,j})$ *is a bicomplex of free* $\mathcal{P}$*-modules, whose total complex is denoted by* $\chi_\bullet(A)$.

Proof. Observe that $\mathcal{P}$ is a quotient algebra of $\mathcal{Q}$ and hence can be viewed as a $\mathcal{Q}$-module. By applying the functor $\mathcal{P}\otimes_{\mathcal{Q}}-$ the bicomplex (2.1) and the natural isomorphism of left $\mathcal{P}$-modules $\mathcal{P}\otimes_{\mathcal{Q}}\mathcal{Q}\otimes(A^i\otimes\wedge^j)\cong\mathcal{P}\otimes A^i\otimes\wedge^j$, we obtain the following bicomplex of Poisson modules:



(3.1)

where $\bar{\eta}^V_{i,j}$ and $\bar{\eta}^H_{i,j}$ are induced by $\mathrm{id}_{\mathcal{P}} \otimes \eta^V_{i,j}$ and $\mathrm{id}_{\mathcal{P}} \otimes \eta^H_{i,j}$, respectively. Erasing all of dashed arrows, we obtain the diagram $C_{\bullet,\bullet}$, and $\delta^V_{0,j} = \varphi_{j-1}\bar{\eta}^V_{1,j-1}$, $\delta^V_{i,j} = \bar{\eta}^V_{i+1,j-1}$ $(i \geqslant 1)$, and $\delta^H_{0,j} = \bar{\eta}_{0,j}$, $\delta^H_{i,j} = \bar{\eta}^H_{i+1,j-1}$ $(i \geqslant 1)$, where

$$\varphi_j((1_A \otimes 1'_A \# \mathbf{1}) \otimes a \otimes \omega^j) = (1_A \otimes 1'_A \# \mathbf{1}) \otimes (a \wedge \omega^j).$$

It is easily seen that $\delta^H_{i-1,j}\delta^H_{i-1,j+1} = 0$, $\delta^V_{i,j}\delta^V_{i+1,j} = 0$, $\delta^V_{i,j}\delta^H_{i+1,j} + \delta^H_{i,j}\delta^V_{i,j+1} = 0$ for all $i \geqslant 1$, and $\delta^V_{0,j}\delta^V_{1,j} = \varphi_{j-1}\bar{\eta}^V_{1,j-1}\bar{\eta}^V_{2,j-1} = 0$. It remains to check $\delta^V_{0,j}\delta^H_{1,j} + \delta^H_{0,j}\delta^V_{0,j+1} = 0$ for all $j \geqslant 1$. We only show the case where $j = 1$, and that the general cases can be proved similarly. By definition, we have

$$\begin{aligned}
&(\delta^V_{0,1}\delta^H_{1,1})((1_A \otimes 1'_A \# \mathbf{1}) \otimes a \otimes b \otimes x)\\
&\quad = \delta^V_{0,1}((1_A \otimes 1'_A \# x) \otimes a \otimes b)\\
&\quad = \delta^V_{0,1}[(1_A \otimes 1'_A \# x)(1_A \otimes 1'_A \# \mathbf{1} \otimes a \otimes b)\\
&\qquad - 1_A \otimes 1'_A \# \mathbf{1} \otimes \{x, a\} \otimes b - 1_A \otimes 1'_A \# \mathbf{1} \otimes a \otimes \{x, b\}]\\
&\quad = (1_A \otimes 1'_A \# x)[a \otimes 1_A \# \mathbf{1} \otimes b - 1_A \otimes 1'_A \# \mathbf{1} \otimes ab + 1_A \otimes b' \# \mathbf{1} \otimes a]\\
&\qquad - \{x, a\} \otimes 1'_A \# \mathbf{1} \otimes b + 1_A \otimes 1'_A \# \mathbf{1} \otimes \{x, a\}b - 1_A \otimes b' \# \mathbf{1} \otimes \{x, a\}\\
&\qquad - a \otimes 1'_A \# \mathbf{1} \otimes \{x, b\} + 1_A \otimes 1'_A \# \mathbf{1} \otimes a\{x, b\} - 1_A \otimes \{x, b\}' \# \mathbf{1} \otimes a\\
&\quad = (a \otimes 1'_A \# x) \otimes b - (1_A \otimes 1'_A \# x) \otimes ab + (1_A \otimes b' \# x) \otimes a + (1_A \otimes 1'_A \# \mathbf{1}) \otimes \{x, a\}b\\
&\qquad - (1_A \otimes b' \# \mathbf{1}) \otimes \{x, a\} + (1_A \otimes 1'_A \# \mathbf{1}) \otimes a\{x, b\} - (1_A \otimes \{x, b\}' \# \mathbf{1}) \otimes a\\
&\quad = -\delta^H_{0,1}((a \otimes 1'_A \# \mathbf{1}) \otimes b \wedge x - (1_A \otimes 1'_A \# \mathbf{1}) \otimes ab \wedge x + (1_A \otimes b' \# \mathbf{1}) \otimes a \wedge x)\\
&\quad = -(\delta^H_{0,1}\delta^V_{0,2})(1_A \otimes 1'_A \# \mathbf{1} \otimes a \otimes b \otimes x),
\end{aligned}$$

which completes the proof. □

Let M be a Poisson module over A. Applying the functor $\mathrm{Hom}_{\mathcal{P}}(-, M)$ to the bicomplex $C_{\bullet,\bullet}$ in Proposition 3.1, one can get a bicomplex $PC^{\bullet,\bullet}(A; M)$.

3.2 FGV-Poisson cohomology groups

Let M be a Poisson module over A. Recall that the n-th FGV-Poisson cohomology groups of A with coefficients in M in [5], denoted by $\mathrm{HP}^n(A, M)$, is defined as the n-th cohomology group of the total complex of the following bicomplex $\widetilde{C}^{\bullet,\bullet}(A; M)$:

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(A^3, M) & \xrightarrow{\delta_H} & \mathrm{Hom}(A^3 \otimes \wedge^1, M) & \xrightarrow{\delta_H} & \mathrm{Hom}(A^3 \otimes \wedge^2, M) & \longrightarrow & \cdots \\
 & & \delta_V \uparrow & & \delta_V \uparrow & & \uparrow \delta_V & & \\
0 & \longrightarrow & \mathrm{Hom}(A^2, M) & \xrightarrow{\delta_H} & \mathrm{Hom}(A^2 \otimes \wedge^1, M) & \xrightarrow{\delta_H} & \mathrm{Hom}(A^2 \otimes \wedge^2, M) & \longrightarrow & \cdots, \\
 & & \delta_v \uparrow & & \delta_v \uparrow & & \uparrow \delta_v & & \\
M & \xrightarrow{\delta_h} & \mathrm{Hom}(\wedge^1, M) & \xrightarrow{\delta_h} & \mathrm{Hom}(\wedge^2, M) & \xrightarrow{\delta_h} & \mathrm{Hom}(\wedge^3, M) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where δ_h is the Chevalley-Eilenberg differential, δ_V is the Hochschild differential, and δ_v is the composition of the natural homomorphism $\mathrm{Hom}(\wedge^j, M) \hookrightarrow \mathrm{Hom}(A \otimes \wedge^{j-1}, M)$ and the Hochschild differential $\mathrm{Hom}(A \otimes \wedge^{j-1}, M) \to \mathrm{Hom}(A^2 \otimes \wedge^{j-1}, M)$, and $\delta_H : \mathrm{Hom}(A^i \otimes \wedge^{j-1}, M) \to \mathrm{Hom}(A^i \otimes \wedge^j, M)$ is given by

$$
\begin{aligned}
&(\delta_H f)(a_1 \otimes \cdots \otimes a_i \otimes (x_1 \wedge \cdots \wedge x_j)) \\
&\quad = \sum_{l=1}^{j} (-1)^{l+1} \bigg(\{x_l, f(a_1 \otimes \cdots \otimes a_i \otimes (x_1 \wedge \cdots \hat{x}_l \cdots \wedge x_j))\}_* \\
&\qquad - \sum_{t=1}^{i} f(a_1 \otimes \cdots \otimes \{x_l, a_t\} \otimes \cdots \otimes a_i \otimes (x_1 \wedge \cdots \hat{x}_l \cdots \wedge x_j)) \bigg) \\
&\qquad + \sum_{1 \leqslant p < q \leqslant j} (-1)^{p+q} f(a_1 \otimes \cdots \otimes a_i \otimes (\{x_p, x_q\} \wedge x_1 \wedge \cdots \hat{x}_p \cdots \hat{x}_q \cdots \wedge x_j)).
\end{aligned}
$$

Remark 3.2. Note that the differential δ_H is essentially a Lie algebra differential. In fact, $\mathrm{Hom}(A^i \otimes \wedge^j, M) \cong \mathrm{Hom}(\wedge^j, \mathrm{Hom}(A^i, M))$ and $\mathrm{Hom}(A^i, M)$ is a Lie module over A with the action given by

$$
\{x, f\}_*(a_1 \otimes \cdots \otimes a_i) = \{x, f(a_1 \otimes \cdots \otimes a_i)\}_* - \sum_{t=1}^{i} f(a_1 \otimes \cdots \{x, a_t\} \cdots \otimes a_i)
$$

for any $f \in \mathrm{Hom}(A^i, M)$ and any $a_1 \otimes \cdots \otimes a_i \in A^i$.

The total complex of $\widetilde{C}^{\bullet,\bullet}(A; M)$ is of the form

$$
\begin{aligned}
0 \to M \xrightarrow{d^0} \mathrm{Hom}(A, M) \xrightarrow{d^1} \mathrm{Hom}(A^2 \oplus \wedge^2, M) \xrightarrow{d^2} \mathrm{Hom}\Bigg(\bigoplus_{\substack{i+j=3 \\ i \neq 1}} A^i \otimes \wedge^j, M \Bigg) \\
\to \cdots \to \mathrm{Hom}\Bigg(\bigoplus_{\substack{i+j=n \\ i \neq 1}} A^i \otimes \wedge^j, M \Bigg) \xrightarrow{d^n} \mathrm{Hom}\Bigg(\bigoplus_{\substack{i+j=n+1 \\ i \neq 1}} A^i \otimes \wedge^j, M \Bigg) \to \cdots,
\end{aligned}
$$

and we call it the *FGV-Poisson complex of A with coefficients in M*. The FGV-Poisson complex with coefficients in A is called the *FGV-Poisson complex of A*, and we simply denote $\mathrm{HP}^n(A; A)$ by $\mathrm{HP}^n(A)$ and call it the n-th *FGV-Poisson cohomology group* of A.

Remark 3.3. The lower-dimensional FGV-Poisson cohomology groups have the following explicit interpretation:

Clearly, $\mathrm{HP}^0(A; M) = \{m \in M \mid \{a, m\}_* = 0, \forall\, a \in A\}$. In particular, $\mathrm{HP}^0(A)$ is the center of the Lie algebra A.

$\mathrm{HP}^1(A; M) = \mathrm{PDer}(A; M)/\mathrm{IPDer}(A; M)$ is the outer Poisson derivations of A with coefficients in M, where $\mathrm{PDer}(A; M)$ is the set of Poisson derivations and $\mathrm{IPDer}(A; M) = \{\{-, m\}_* \mid m \in M\}$ is the set of inner ones. Recall that a linear map from A to M is called a *Poisson derivation* if it is simultaneously a derivation in the Lie algebra sense and in the associative sense.

For any $f = (f_1, f_0) \in \mathrm{Ker} d^2$, we may define a new Poisson algebra $A \ltimes_f M$, which is called the *extension of A by M along f*. As a $\Bbbk$-vector space, $A \ltimes_f M = A \oplus M$; and the associative multiplication and the Lie bracket are given by

$$
\begin{aligned}
&(a, x) \cdot (a', x') = (aa', ax' + xa' + f_1(a \otimes a')), \\
&\{(a, x), (a', x')\} = (\{a, a'\}, \{a, x'\}_* - \{a', x\}_* + f_0(a \wedge a')\}.
\end{aligned}
$$

We have the following standard result; compare also with the deformation theory.

Proposition 3.4. *For any $f \in \mathrm{Ker} d^2$, $A \ltimes_f M$ is a Poisson algebra. Moreover, for any $f, g \in \mathrm{Ker} d^2$ with $\overline{f} = \overline{g}$ in $\mathrm{HP}^2(A, M)$, there is an isomorphism of Poisson algebras $A \ltimes_f M \simeq A \ltimes_g M$.*

Applying the complex $\chi_\bullet(A)$, we can interpret the FGV-Poisson cohomology of A by the derived functor $\mathrm{RHom}_{\mathcal{P}}(\chi_\bullet(A), -)$.

Theorem 3.5. *Let A be a Poisson algebra and M be a Poisson module over A. Then*

$$
\mathrm{HP}^n(A, M) \cong \mathrm{H}^n \mathrm{Hom}_{\mathcal{P}}(\chi_\bullet(A), M).
$$

Proof. By identifying $\mathrm{Hom}_{\mathcal{P}}(\mathcal{P}\otimes A^i\otimes\wedge^j, M)$ with $\mathrm{Hom}(A^i\otimes\wedge^j, M)$, we easily deduce that the bicomplex $PC^{\bullet,\bullet}(A;M)$ is isomorphic to $\widetilde{C}^{\bullet,\bullet}(A;M)$. The rest of the proof is obvious. □

From now on, we identify the bicomplex

$$PC^{\bullet,\bullet}(A;M)=\mathrm{Hom}_{\mathcal{P}}(C_{\bullet,\bullet},M)$$

with the bicomplex $\widetilde{C}^{\bullet,\bullet}(A,M)$, called the *FGV-Poisson bicomplex* of A with coefficients in M, whose total complex is just the FGV-Poisson complex.

The theorem says that the FGV-Poisson cohomology is essentially defined by derived functors. An advantage is that the well-developed results and tools in the usual cohomology theory for algebras apply for the FGV-Poisson cohomology theory as well. For example, we may obtain the following long exact sequence of Poisson cohomology groups from a short exact sequence of Poisson modules, which is not so obvious from the original definition given by Flato et al. [5].

Proposition 3.6. *Let A be a Poisson algebra, and $0\to M_1\to M_2\to M_3\to 0$ be an exact sequence of Poisson A-modules. Then we have a long exact sequence*

$$\begin{aligned}0&\to \mathrm{HP}^0(A,M_1)\to \mathrm{HP}^0(A,M_2)\to \mathrm{HP}^0(A,M_3)\to \mathrm{HP}^1(A,M_1)\to\cdots\\ &\to \mathrm{HP}^n(A,M_1)\to \mathrm{HP}^n(A,M_2)\to \mathrm{HP}^n(A,M_3)\to \mathrm{HP}^{n+1}(A,M_1)\to\cdots\end{aligned}$$

of FGV-Poisson cohomology groups.

Proof. Let $\mathcal{P}$ be the Poisson enveloping algebra of A. Then each Poisson module can be viewed as a $\mathcal{P}$-module, and the sequence $0\to M_1\to M_2\to M_3\to 0$ gives a distinguished triangle in the derived category $D(\mathcal{P})$, where each M_i is regarded as a stalk complex concentrated in degree 0. Applying the cohomology functor $\mathrm{Hom}_{D(\mathcal{P})}(\chi_\bullet(A),-)$, we obtain the desired long exact sequence. □

4 Formal deformations and deformation quantization

In [5, Section 6], a deformation theory for Poisson algebras has been developed. It is worth pointing out that, even for commutative Poisson algebras, the noncommutative version of the deformation theory is quite useful. In fact, Kontsevich's deformation quantization is explained as a special case of formal deformations in the noncommutative sense, and by using the Poisson cohomology group we can give some necessary condition for the existence of the deformation quantization.

4.1 Formal Poisson deformations

Let $(A,\cdot,\{-,-\})$ be a Poisson algebra over $\Bbbk$. Let $\Bbbk[[t]]$ and $A[[t]]$ be the formal power series ring in one variable t with coefficients in $\Bbbk$ and A, respectively. A *formal deformation* of the Poisson algebra A means a $\Bbbk[[t]]$-Poisson algebra $(A[[t]],m_t,l_t)$ such that $(A[[t]],m_t)$ and $(A[[t]],l_t)$ are the formal deformations of the associative algebra $(A,\cdot)$ and the Lie algebra $(A,\{-,-\})$, respectively. Clearly the associative multiplication m_t and the Lie bracket l_t are determined by their restrictions to the subset A. We may write

$$\begin{aligned}m_t(a,b)&=m_0(a,b)+tm_1(a,b)+t^2m_2(a,b)+\cdots,\\ l_t(a,b)&=l_0(a,b)+tl_1(a,b)+t^2l_2(a,b)+\cdots\end{aligned}$$

for any $a,b\in A$, where $m_i,l_i\colon A\times A\to A$ are $\Bbbk$-bilinear maps. By definition, $m_0(a,b)=ab$ and $l_0(a,b)=\{a,b\}$.

As we expected, the FGV-Poisson cohomology controls the deformation of Poisson algebra, which essentially goes to Flato et al. [5, Section 6]. In fact, the associativity, the Leibniz rule and the Jacobi identity read as

$$\sum_{\substack{p+q=n\\ p,q>0}}(m_p(m_q(a,b),c)-m_p(a,m_q(b,c)))=am_n(b,c)-m_n(ab,c)+m_n(a,bc)-m_n(a,b)c, \qquad (\mathrm{D}_n\text{-}1)$$

$$\sum_{\substack{p+q=n\\p,q>0}}(l_q(m_p(a,b),c)-m_p(a,l_q(b,c))-m_p(l_q(a,c),b))$$
$$=al_n(b,c)-l_n(ab,c)+l_n(a,c)b+\{c,m_n(a,b)\}-m_n(a,\{c,a\})-m_n(\{c,a\},b), \qquad (\mathrm{D}_n\text{-}2)$$
$$\sum_{\substack{p+q=n\\p,q>0}}[l_q(l_p(a,b),c)+l_q(l_p(b,c),a)+l_q(l_p(c,a),b)]$$
$$=l_n(a,\{b,c\})+l_n(b,\{c,a\})+l_n(c,\{a,b\})-\{l_n(a,b),c\}-\{l_n(b,c),a\}-\{l_n(c,a),b\} \qquad (\mathrm{D}_n\text{-}3)$$

for all $a,b,c\in A$ and $n\geqslant 1$. (D_n-1), (D_n-2) and (D_n-3) are called the *deformation equations* for Poisson algebras.

By definition, $m_0(a,b)=ab$ and $l_0(a,b)=\{a,b\}$. The pair (m_1,l_1) is called an *infinitesimal deformation* of A, which gives a Poisson structure on the quotient space $A[[t]]/(t^2)$.

We set $F'_n,F''_n,F'''_n\colon A^3\to A$ to be the functions given by the left-hand sides of the deformation equations (D_n-1), (D_n-2) and (D_n-3), respectively.

Similar to the associative algebra case, deformations of Poisson algebras will meet some *obstructions* in the FGV-Poisson cohomology, which are all deducible from the deformation equations. By a routine check, we have the following result.

Theorem 4.1 (See [5, Section 6]). *Let $(A,\cdot,\{-,-\})$ be a Poisson algebra and $\mathrm{HP}^n(A)$ be the n-th FGV-Poisson cohomology group of A.*

(i) *The pair (m_1,l_1) is an infinitesimal deformation of A if and only if (m_1,l_1) is a 2-cocycle in the Poisson complex of A. In particular, if $\mathrm{HP}^2(A)=0$, then A has no nontrivial formal deformation.*

(ii) *For each $n\geqslant 2$, if (D_i-1), (D_i-2) and (D_i-3) hold for $i=1,\ldots,n-1$, then (F'_n,F''_n,F'''_n) is a 3-cocycle, and hence $\mathrm{HP}^3(A)$ is the "obstruction cocycle". In particular, if $\mathrm{HP}^3(A)=0$, then all the obstructions vanish.*

4.2 Deformation quantization

In this subsection, we always assume that $\Bbbk$ is an algebraically closed field and A is a commutative associative algebra. We will explain in the sequel how to understand Kontsevich's deformation quantization as a special case of the formal deformation of Poisson algebras.

We first recall the definition of the deformation quantization. For this, we need the following observation.

Proposition 4.2. *Let A be a commutative algebra over $\Bbbk$ and $(A[[t]],m_t)$ be a formal deformation of A. If for some positive integer n, $m_i(a,b)=m_i(b,a)$ for any $a,b\in A$ and any $0\leqslant i\leqslant n-1$, then $\{a,b\}=m_n(a,b)-m_n(b,a)$ gives a Poisson structure on A.*

Proof. By assumption, $[A,A]_t\subseteq t^nA[[t]]$, and thus we may consider a standard Poisson structure $l_t=\frac{1}{t^n}[-,-]_t$ on $A[[t]]$, where $[-,-]_t$ is the commutator of m_t. The bracket l_t can be expressed as $l_t(a,b)=\sum_{i\geqslant 0}l_i(a,b)t^i$ with each

$$l_i(a,b)=m_{n+i}(a,b)-m_{n+i}(b,a)$$

for any $a,b\in A$. By comparing the constant terms of two sides of the following equations:

$$\begin{aligned}&l_t(a,b)+l_t(b,a)=0,\\&l_t(l_t(a,b),c)+l_t(l_t(b,c),a)+l_t(l_t(c,a),b)=0,\\&l_t(m_t(a,b),c)=m_t(a,l_t(b,c))+m_t(l_t(a,b),c),\end{aligned}$$

we get

$$\begin{aligned}&l_0(a,b)+l_0(b,a)=0,\\&l_0(l_0(a,b),c)+l_0(l_0(b,c),a)+l_0(l_0(c,a),b)=0,\\&l_0(ab,c)=al_0(b,c)+l_0(a,b)c,\end{aligned}$$

and hence l_0 gives a Poisson structure on A. □

Definition 4.3. Let $P = (A, \cdot, \{-,-\})$ be a Poisson algebra, which is commutative as an associative algebra. If there is a formal deformation $(A[[t]], m_t)$ of A as an associative algebra and a positive integer n, such that $m_i(a,b) = m_i(b,a)$ for $i = 1, \ldots, n-1$ and

$$\{a,b\} = m_n(a,b) - m_n(b,a)$$

for all $a, b \in A$, then we say that $(A[[t]], m_t)$ is an *n-deformation quantization* of P.

By definition, a 1-deformation quantization $(A[[t]], m_t)$ is exactly a *deformation quantization* of P in the sense of Kontsevich [10]. In this case, P is said to be the *classical limit* of m_t (see [3, Definition 8.4]). For more details, we refer to [3, 10, 16].

Theorem 4.4. *Let $P = (A, \cdot, \{-,-\})$ be a commutative Poisson algebra.*

(1) *If P admits an n-deformation quantization for some $n \geqslant 1$, then P can be deformed to a standard Poisson algebra.*

(2) *Assume further that each formal deformation $(A[[t]], m_t)$ of $(A, \cdot)$ has only standard Poisson structures. Then P has an n-deformation quantization for some positive integer n if and only if P has a formal deformation.*

Proof. (1) Suppose that the Poisson algebra $(A, \cdot, \{-,-\})$ has an n-deformation quantization $(A[[t]], m_t)$ satisfying $m_i(a,b) = m_i(b,a)$ for any $a, b \in A$ and $i = 0, 1, \ldots, n-1$. By definition, we have

$$\{a,b\} = m_n(a,b) - m_n(b,a)$$

for all $a, b \in A$. We consider the standard Poisson algebra $(A[[t]], m_t, l_t)$ with

$$l_t(a,b) = \frac{1}{t^n}(m_t(a,b) - m_t(b,a)).$$

Since $ab = m_0(a,b)$ and $\{a,b\} = m_n(a,b) - m_n(b,a) = l_0(a,b)$, the standard Poisson algebra $(A[[t]], m_t, l_t)$ is just the formal deformation of $(A, \cdot, \{-,-\})$.

(2) By (1), we only need to prove the part of "if". Suppose that $(A, \cdot, \{-,-\})$ has a formal deformation $(A[[t]], m_t, l_t)$, where

$$m_t(a,b) = \sum_{i=0}^{\infty} m_i(a,b)t^i \quad \text{and} \quad l_t(a,b) = \sum_{i=0}^{\infty} l_i(a,b)t^i.$$

Since $(A[[t]], m_t)$ has only standard Poisson structures, there exists some $\lambda(t), \mu(t) \in \Bbbk[[t]]$ such that

$$l_t(a,b) = \frac{\mu(t)}{\lambda(t)}(m_t(a,b) - m_t(b,a)).$$

Without loss of generality, we assume that $\lambda(t) = t^n$ for some $n \geqslant 1$, and

$$\mu(t) = \mu_0 + \mu_1 t + \cdots + \mu_i t^i + \cdots \in \Bbbk[[t]]$$

with $\mu_i \in \Bbbk, i \geqslant 0$ and $\mu_0 \neq 0$.

By comparing the both sides of the above expression of $l_t(a,b)$, we obtain that $m_i(a,b) = m_i(b,a)$ for $i = 0, 1, \ldots, n-1$ and

$$\{a,b\} = l_0(a,b) = \mu_0(m_n(a,b) - m_n(b,a)).$$

Observe that for any $\nu \in \Bbbk$, the family of $\Bbbk$-bilinear maps $m_i'(a,b) = \nu^i m_i(a,b)$ $(i \geqslant 0)$ forms a formal deformation of A. So we can choose $\nu = \mu_0^{\frac{1}{n}}$ since $\Bbbk$ is algebraically closed. It follows that $\{a,b\} = m_n'(a,b) - m_n'(b,a)$ for any $a, b \in A$ and therefore A has an n-deformation quantization. □

Example 4.5. Let A be a Poisson algebra which is an integral domain as an algebra. Then A has an n-deformation quantization for some positive integer n if and only if A has a formal deformation. In fact, any formal deformation $(A[[t]], m_t)$ of an integral domain A is prime, and hence has only standard Poisson structures by a result of Farkas and Letzter (see [4, Theorem 1.2] for more details). Then Theorem 4.4 does work.

By Theorem 4.4(1), we can give a necessary condition of the existence of the deformation quantization of a Poisson algebra.

Proposition 4.6. *Let $P = (A, \cdot, \{-,-\})$ be a nontrivial commutative Poisson algebra. If* $\mathrm{HP}^2(A) = 0$, *then P has no deformation quantization.*

Proof. Assume that $(A[[t]], m_t)$ is a deformation quantization of P. By Theorem 4.4(1), $(A[[t]], m_t, l_t)$ is a formal deformation of $(A, \cdot, \{-,-\})$, where

$$l_t(a,b) = \frac{1}{t}(m_t(a,b) - m_t(b,a))$$

for all $a, b \in A$. Since $\mathrm{HP}^2(A) = 0$, by Theorem 4.1(i), the formal deformation $(A[[t]], m_t, l_t)$ is equivalent to the trivial one, i.e., there exists an isomorphism

$$g\colon (A[[t]], m_t, l_t) \to (A[[t]], m'_t, l'_t)$$

of Poisson algebras over $\Bbbk[[t]]$ such that $g(a) \in a + tA[[t]]$ for any $a \in A$, where $m'_t(a,b) = ab$ and $l'_t(a,b) = \{a,b\}$ for $a, b \in A$. Therefore,

$$\begin{aligned}\{a,b\} &= l'_t(a,b) = g(l_t(g^{-1}(a), g^{-1}(b)))\\ &= g\frac{1}{t}(m_t(g^{-1}(a), g^{-1}(b)) - m_t(g^{-1}(b), g^{-1}(a)))\\ &= \frac{1}{t}(m'_t(a,b) - m'_t(b,a)) = \frac{1}{t}(ab - ba) = 0,\end{aligned}$$

which leads to a contradiction. □

5 Comparing FGV-Poisson and LP cohomologies

Recall the definition of the Lichnerowicz-Poisson cohomology for a commutative Poisson algebra (see [6, 8, 11, 13, 16] for detail). We set $\xi^i(A) = 0$ for any $i < 0$, $\xi^0(A) = A$ and $\xi^1(A) = \mathrm{Der}(A)$. For $i \geqslant 2$, let $\xi^\bullet(A)$ be the subspace of $\mathrm{Hom}(\wedge^\bullet, A)$ consisting of all the skew-symmetric multiderivations of A. The subcomplex $(\xi^\bullet(A), d)$ of the Chevalley-Eilenberg complex $(\mathrm{Hom}(\wedge^\bullet, A), d)$ is called the *Lichnerowicz-Poisson complex of A*, or simply *LP-complex*, and its n-th cohomology group, denoted by $\mathrm{H}^n_{LP}(A)$, is called the *n-th Lichnerowicz-Poisson* (*LP for short*) *cohomology group of A* (see [13]).

Remark 5.1. Let A be a Poisson algebra. Then by definition we have

$$\mathrm{HP}^0(A) = \mathrm{H}^0_{LP}(A), \quad \mathrm{HP}^1(A) = \mathrm{H}^1_{LP}(A).$$

In fact, an easy calculation shows that the 0-th LP-cohomology $\mathrm{H}^0_{LP}(A)$ is the center of the Lie algebra A, and $\mathrm{H}^1_{LP}(A)$ is exactly the set of Poisson derivations.

Moreover, by a direct calculation, we show that the LP-cohomology relates to the FGV-Poisson cohomology closely.

We simply denote the FGV-Poisson bicomplex $PC^{\bullet,\bullet}(A,A)$ by $PC^{\bullet,\bullet}$. Considering the spectral sequence induced by the second filtration of the bicomplex $PC^{\bullet,\bullet}$, we obtain a complex

$$0 \to A \to \mathrm{H}^0_{\mathrm{I}}(PC^{\bullet,1}) \to \mathrm{H}^0_{\mathrm{I}}(PC^{\bullet,2}) \to \cdots \to \mathrm{H}^0_{\mathrm{I}}(PC^{\bullet,j}) \to \mathrm{H}^0_{\mathrm{I}}(PC^{\bullet,j+1}) \to \cdots,$$

where

$$\mathrm{H}^0_{\mathrm{I}}(PC^{\bullet,j}) = \{f \in \mathrm{Hom}(\wedge^j, A) \mid \delta_v(f) = 0\}$$

is the space of all the skew-symmetric n-fold derivations of A. On the spectral sequence given by the first and second filtrations of a bicomplex, we refer to [2, Chapter XV, Section 6].

Proposition 5.2. *Keep the above notations. We have*

$$\mathrm{H}^n_{LP}(A) \cong E^{0,n}_{\mathrm{II}\,2},$$

where $E^{0,n}_{\mathrm{II}2}$ *is the* $(n,0)$*-term in the spectral sequence given by the second filtration of the FGV-Poisson bicomplex.*

Remark 5.3. We would like to mention that the FGV-Poisson cohomology controls the formal deformations of a Poisson algebra (see Subsection 4.1), while the LP-cohomology controls those deformations which only deform the Lie bracket.

To study the further relation between the FGV-Poisson cohomology and the LP-cohomology, we consider the first filtration of the FGV-Poisson bicomplex, whose $(n,0)$-term is given by

$$\zeta^1(A) = \{f \in \mathrm{Hom}_{\Bbbk}(A,A) \mid \{x, f(y)\} + \{y, f(x)\} - f(\{x,y\}) = 0, \text{ for all } x, y \in A\},$$

$$\zeta^n(A) = \left\{ f \in \mathrm{Hom}_{\Bbbk}(A^n, A) \,\middle|\, \{x, f(a_1, \ldots, a_n)\} = \sum_{i=1}^{n} f(a_1, \ldots, \{x, a_i\}, \ldots, a_n) \right. \left. \text{for all } x, a_i \in A,\ i = 1, \ldots, n \right\}$$

for $n \geqslant 2$.

The complex $\zeta^\bullet$ is a subcomplex of the Hochschild complex. We call $\zeta^\bullet$ the *Hochschild-Poisson complex*, and its n-th cohomology $\mathrm{H}^n_{HP}(A)$ the *n-Hochschild-Poisson cohomology group*.

Proposition 5.4. *Let* A *be a commutative Poisson algebra over* $\Bbbk$*, which is finitely generated and smooth as an associative algebra. Then there is an isomorphism*

$$\mathrm{HP}^2(A) \cong \mathrm{H}^2_{HP}(A) \oplus \mathrm{H}^2_{LP}(A).$$

Proof. Let $\alpha \in \mathrm{Hom}_{\Bbbk}(A^2, A)$ and $\beta \in \mathrm{Hom}_{\Bbbk}(\wedge^2, A)$ such that (α, β) is an FGV-Poisson 2-cocycle. By definition, α is a Hochschild 2-cocycle. Since A is finitely generated and smooth, by the Hochschild-Kostant-Rosenberg (HKR) theorem we have $\mathrm{HH}^2(A) = \bigwedge^2_A \mathrm{Der}(A)$ (see [7]). Without loss of generality, we may assume that

$$\alpha(f,g) = a(\delta_1(f)\delta_2(g) - \delta_1(g)\delta_2(f))$$

for some $a \in A$ and $\delta_1, \delta_2 \in \mathrm{Der}(A)$. By the definition of the FGV-Poisson cohomology, we have

$$f\beta(g,h) - \beta(fg,h) + \beta(f,h)g = \alpha(\{h,f\},g) + \alpha(f,\{h,g\}) - \{h, \alpha(f,g)\}$$

for any $f, g, h \in A$. Note that for f and g, the left-hand side is symmetric and the right one is anti-symmetric. It forces that both sides equal 0, which means that α is a Hochschild-Poisson 2-cocycle and β is a Lichnerowicz-Poisson 2-cocycle.

On the other hand, if (α, β) is an FGV-Poisson 2-coboundary, then α is a Hochschild-Poisson 2-coboundary and β is a Lichnerowicz-Poisson 2-coboundary. Then we obtain a well-defined linear map

$$\Psi\colon \mathrm{HP}^2(A) \to \mathrm{H}^2_{HP}(A) \oplus \mathrm{H}^2_{LP}(A), \quad \overline{(\alpha,\beta)} \mapsto (\overline{\alpha}, \overline{\beta}).$$

Conversely, let α be a Hochschild-Poisson 2-cocycle and β be a Lichnerowicz-Poisson 2-cocycle. Clearly, (α, β) is an FGV-Poisson 2-cocycle. If moreover, α is a Hochschild-Poisson 2-coboundary, and β is a Lichnerowicz-Poisson 2-coboundary, then there exist a Lie derivation φ_1 and a derivation φ_2 such that

$$\alpha(f,g) = f\varphi_1(g) - \varphi_1(fg) + \varphi_1(f)g,$$
$$\beta(f,g) = \{\varphi_2(f), g\} + \{f, \varphi_2(g)\} - \varphi_2(\{f,g\}).$$

It is easily seen that $(\alpha, \beta) \in \mathrm{Hom}_{\Bbbk}(A^2 \oplus \wedge^2, A)$ is an FGV-2-coboundary given by $\varphi_1 + \varphi_2$. Thus we obtain a map

$$\mathrm{H}^2_{HP}(A) \oplus \mathrm{H}^2_{LP}(A) \to \mathrm{HP}^2(A), \quad (\overline{\alpha}, \overline{\beta}) \mapsto \overline{(\alpha,\beta)},$$

which is converse to Ψ, and the desired isomorphism follows. □

Example 5.5. Let $A = \Bbbk[x, y]$ be a polynomial algebra in two variables x and y with the Poisson bracket given by $\{-,-\} = a_0\partial_x \wedge \partial_y$ for some $a_0 \in A$, where ∂_x and ∂_y are the partial derivatives with respect to the variables x and y, respectively. This bracket was the original Poisson bracket studied by many people including Poisson [14] when $\Bbbk = \mathbb{R}$.

First by the HKR Theorem [7], $\mathrm{HH}^2(A) = A(\partial_x \wedge \partial_y)$. Then an easy calculation shows that the 2nd Hochschild-Poisson cohomology group is $\Bbbk(\overline{\partial_x \wedge \partial_y})$. Applying the duality between the LP-cohomology and the LP-homology (see [9, Equation (7.2)] or [12, Remark 3.6]), we know that the 2nd LP-cohomology group vanishes. Therefore, by Proposition 5.4, we get $\mathrm{HP}^2(A) \cong \Bbbk$.

Next, we consider the formal deformation and the deformation quantization of A. It is direct to show that A has a deformation quantization $(A[[t]], m_t)$ with the star product m_t given by

$$m_t(f, g) = \sum_{n=0}^{\infty} m_n(f, g)t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\partial_x^n(f)\partial_y^n(g)$$

for all $f, g \in A \subset A[[t]]$. Furthermore, A has a formal deformation $(A[[t]], m_t, l_t)$ in the noncommutative sense, where

$$\begin{aligned} l_t(f, g) &= \sum_{n=0}^{\infty} l_n(f, g)t^n = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}(\partial_x^{n+1}(f)\partial_y^{n+1}(g) - \partial_x^{n+1}(g)\partial_y^{n+1}(f)) \\ &= \frac{1}{t}(m_t(f, g) - m_t(g, f)) \end{aligned}$$

for all $f, g \in A$. Since A is integral domain, we know that $(A[[t]], m_t)$ is a prime algebra. By Example 4.4 and Proposition 4.3, this formal deformation exactly gives the above deformation quantization.

6 Connection with quasi-Poisson cohomology groups

In Section 3, we have introduced the complex $\chi_\bullet(A)$ for a Poisson algebra A and applied it to study the Poisson cohomology. However, whether $\chi_\bullet(A)$ is quasi-isomorphic to a stalk complex of some Poisson module, is not known yet. We will make some discussion on this question.

Let $\mathcal{Q}$ be the quasi-Poisson enveloping algebra of $(A, \cdot, \{-,-\})$ and $\Omega^2(A)$ be the second syzygy of A as an A^e-module. More precisely, $\Omega^2(A)$ is the quotient module of A^e-module A^4 modulo the submodule generated by

$$\{a \otimes b \otimes c \otimes 1_A - 1_A \otimes ab \otimes c \otimes 1_A + 1 \otimes a \otimes bc \otimes 1_A - 1 \otimes a \otimes b \otimes c \mid a, b, c \in A\}.$$

Note that $\Omega^2(A)$ is also a quotient module of A^4 as a $\mathcal{Q}$-module. Similar to the construction of the free $\mathcal{Q}$-resolution of A in [1], we consider the bar resolution of the A^e-module $\Omega^2(A)$

$$\mathbb{S}_\bullet\colon \cdots \to A^{i+4} \to A^{i+3} \to \cdots \to A^5 \to A^4 \to 0$$

of $\Omega^2(A)$, and the free resolution of $\Bbbk$ as a trivial $\mathcal{U}(A)$-module (see [2, Chapter XIII, Section 7])

$$\mathbb{K}_\bullet\colon \cdots \to \mathcal{U}(A) \otimes \wedge^j \to \mathcal{U}(A) \otimes \wedge^{j-1} \to \cdots \to \mathcal{U}(A) \otimes \wedge^1 \to \mathcal{U}(A) \to 0.$$

By taking the total complex of the tensor product of $\mathbb{S}_\bullet$ and $\mathbb{K}_\bullet$, we obtain a free resolution of the $\mathcal{Q}$-module $\Omega^2(A)$

$$\begin{aligned} \mathbb{T}'_\bullet\colon \cdots \to \bigoplus_{i+j=n} A^{i+4} \otimes \mathcal{U}(A) \otimes \wedge^j \to \bigoplus_{i+j=n-1} A^{i+4} \otimes \mathcal{U}(A) \otimes \wedge^j \to \cdots \\ \to A^5 \otimes \mathcal{U}(A) \oplus A^4 \otimes \mathcal{U}(A) \otimes \wedge^1 \to A^4 \otimes \mathcal{U}(A) \to 0. \end{aligned}$$

On the other hand, we may apply the functor

$$\mathcal{Q} \underset{\mathcal{U}(A)}{\otimes} -$$

to the resolution $\mathbb{K}_\bullet$, and obtain a free resolution of the $\mathcal{Q}$-module A^2

$$\mathbb{T}''_\bullet\colon \cdots \to \mathcal{Q}\otimes \wedge^n \to \mathcal{Q}\otimes \wedge^{n-1} \to \cdots \to \mathcal{Q} \to 0.$$

By the definition of the bicomplex $C_{\bullet,\bullet}$ in Proposition 3.1, we immediately get a short exact sequence of complexes

$$0 \to \mathcal{P} \underset{\mathcal{Q}}{\otimes} \mathbb{T}''_\bullet \to \chi_\bullet(A) \to \mathcal{P} \underset{\mathcal{Q}}{\otimes} \mathbb{T}'[1]_\bullet \to 0,$$

where

$$\mathbb{T}'[1]_n = \mathbb{T}'_{n-1} = \bigoplus_{i+j=n-1} A^{i+4} \otimes \mathcal{U}(A) \otimes \wedge^j.$$

Thus we have the following long exact sequence, relating the homologies of the characteristic complex with some torsion groups of quasi-Poisson modules.

Proposition 6.1. *Keep the above notations. Then there is a long exact sequence*

$$\begin{aligned}\cdots \to \mathrm{Tor}_n^{\mathcal{Q}}(\mathcal{P}, A^2) \to \mathrm{H}_n(\chi_\bullet(A)) \to \mathrm{Tor}_{n-2}^{\mathcal{Q}}(\mathcal{P}, \Omega^2(A)) \to \mathrm{Tor}_{n-1}^{\mathcal{Q}}(\mathcal{P}, A^2) \to \cdots \\ \to \mathrm{Tor}_1^{\mathcal{Q}}(\mathcal{P}, A^2) \to \mathrm{H}_1(\chi_\bullet(A)) \to 0 \to \mathcal{P}\otimes A^2 \to \mathrm{H}_0(\chi_\bullet(A)) \to 0.\end{aligned}$$

Now we turn to the calculation of Poisson cohomology groups. Let $QC^\bullet(\Omega^2(A), M)$ be the complex

$$\begin{aligned} 0 &\to \mathrm{Hom}(A^2, M) \to \mathrm{Hom}(A^3 \oplus A^2\otimes \wedge^1, M) \\ &\to \cdots \to \mathrm{Hom}\bigg(\bigoplus_{i+j=n-1} A^{i+2}\otimes \wedge^j, M\bigg) \\ &\to \mathrm{Hom}\bigg(\bigoplus_{i+j=n} A^{i+2}\otimes \wedge^j, M\bigg) \to \cdots, \end{aligned}$$

which is obtained by applying the functor $\mathrm{Hom}_{\mathcal{Q}}(-, M)$ to the resolution $\mathbb{T}'_\bullet$. We deduce the following easy result.

Lemma 6.2. *Let M be a quasi-Poisson module over A. Then*

$$\mathrm{Ext}_{\mathcal{Q}}^n(\Omega^2(A), M) \cong \mathrm{H}^n(QC^\bullet(\Omega^2(A), M)), \quad \forall\, n \geqslant 0.$$

Using the projective resolution $\mathbb{T}''_\bullet$, we have the following lemma.

Lemma 6.3. *Let M be a quasi-Poisson module. Then*

$$\mathrm{Ext}_{\mathcal{Q}}^n(A^2, M) \cong \mathrm{HL}^n(A, M),$$

where $\mathrm{HL}^n(A, M)$ is the n-th Lie algebra cohomology of the Lie algebra $(A, \{-,-\})$ with coefficients in M.

Proof. Here, we use the fact that A^2 is a free A^e-module. In fact, applying the isomorphisms

$$\mathrm{Hom}_{\mathcal{Q}}(A^2 \otimes \mathcal{U}(A) \otimes \wedge^n, M) \cong \mathrm{Hom}(\wedge^n, M),$$

we know that $\mathrm{Hom}_{\mathcal{Q}}(\mathbb{T}''_\bullet, M)$ is isomorphic to the Chevalley-Eilenberg complex

$$CE^\bullet : 0 \to M \to \mathrm{Hom}(\wedge^1, M) \to \mathrm{Hom}(\wedge^2, M) \to \cdots \to \mathrm{Hom}(\wedge^n, M) \to \cdots,$$

and the desired isomorphism follows. □

Let $PC^\bullet(A, M)$ denote the Poisson complex of A with coefficients in M. Clearly we have a short exact sequence of complexes

$$0 \to QC^\bullet(\Omega^2(A), M)[-2] \to PC^\bullet(A, M) \to CE^\bullet \to 0,$$

and combining Lemmas 6.2 and 6.3, we obtain the following long exact sequence.

Theorem 6.4. *Let A be a Poisson algebra and M be a Poisson module over A. Then there exists a long exact sequence*

$$\begin{aligned} 0 &\to \mathrm{HP}^0(A,M) \to \mathrm{HL}^0(A,M) \to 0 \to \mathrm{HP}^1(A,M) \to \mathrm{HL}^1(A,M) \\ &\to \mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A),M) \to \mathrm{HP}^2(A,M) \to \mathrm{HL}^2(A,M) \to \cdots \to \mathrm{Ext}_{\mathcal{Q}}^{n-2}(\Omega^2(A),M) \\ &\to \mathrm{HP}^n(A,M) \to \mathrm{HL}^n(A,M) \to \mathrm{Ext}_{\mathcal{Q}}^{n-1}(\Omega^2(A),M) \to \cdots . \end{aligned} \tag{6.1}$$

We recall a useful spectral sequence introduced in our previous work [1].

Proposition 6.5 (See [1, Lemma 5.2])**.** *Let M and N be modules over $\mathcal{Q}$. Then we have a spectral sequence*

$$\mathrm{Ext}^q_{\mathcal{U}(A)}(\Bbbk, \mathrm{Ext}^p_{A^e}(M,N)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{Q}}(M,N).$$

In conclusion, the above results provide us a way to read the information of the Poisson cohomology from the Lie algebra cohomology and the quasi-Poisson cohomology.

Example 6.6. Let A be the $\Bbbk$-algebra of upper triangular 2×2 matrices. This algebra is known to be the path algebra of the quiver of $\mathbb{A}_2$ type.

Consider the standard Poisson algebra. Clearly, A is a hereditary algebra as an associative algebra and hence $\mathrm{HH}^n(A)=0$ for all $n \geqslant 1$. By a direct computation, we have

$$\mathrm{HL}^0(A)=\Bbbk, \quad \mathrm{HL}^1(A)=\Bbbk^2, \quad \mathrm{HL}^2(A)=\Bbbk,$$

and $\mathrm{HL}^n(A)=0$ for all $n\geqslant 3$, where $\mathrm{HL}^i(A)=\mathrm{Ext}^i_{\mathcal{U}(A)}(\Bbbk,A)$ for each i.

Applying the above spectral sequence and by some direct calculations, we know that

$$\mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A),A)=\Bbbk^3, \quad \mathrm{Ext}_{\mathcal{Q}}(\Omega^2(A),A)=\Bbbk^6, \quad \mathrm{Ext}^2_{\mathcal{Q}}(\Omega^2(A),A)=\Bbbk^3,$$

and $\mathrm{Ext}^n_{\mathcal{Q}}(\Omega^2(A),A)=0$ for all $n\geqslant 3$. Now it follows from Theorem 6.4 that

$$\mathrm{HP}^0(A)=\Bbbk, \quad \mathrm{HP}^1(A)=0, \quad \mathrm{HP}^2(A)=\Bbbk, \quad \mathrm{HP}^3(A)=\Bbbk^5, \quad \mathrm{HP}^4(A)=\Bbbk^3,$$

and $\mathrm{HP}^n(A)=0$ for all $n\geqslant 5$.

7 A further example: $\mathbb{M}_2(\Bbbk)$

Let $A=\mathbb{M}_2(\Bbbk)$ be the standard Poisson algebra, where $\mathbb{M}_2(\Bbbk)$ is the algebra of 2×2 matrices with entries in $\Bbbk$. Clearly, as a Lie algebra, $A=\Bbbk\cdot 1_A \oplus \mathfrak{sl}_2(\Bbbk)$, where $\Bbbk\cdot 1_A$ is an abelian Lie algebra of dimension 1 and $\mathfrak{sl}_2(\Bbbk)$ is the special linear Lie algebra with the standard basis

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Consider the module $\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk)$ over the Lie algebra $\mathfrak{sl}_2(\Bbbk)$. It is easy to show the following decomposition as Lie modules:

$$\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk)=V_5\oplus V_3\oplus V_1,$$

where V_5, V_3 and V_1 are simple modules of dimensions 5, 3 and 1, respectively. More precisely, V_5 has a $\Bbbk$-basis

$$\{e\otimes e, h\otimes e+e\otimes h, e\otimes f-h\otimes h+f\otimes e, f\otimes f, h\otimes f+f\otimes h\},$$

V_3 has a $\Bbbk$-basis

$$\{h\otimes e-e\otimes h, e\otimes f-f\otimes e, h\otimes f-f\otimes h\},$$

and V_1 has a $\Bbbk$-basis

$$\{2e\otimes f+h\otimes h+2f\otimes e\},$$

where all the basis elements are given by weight vectors.

By some direct computations, we have the following well-known fact which will be useful in our later calculation.

Lemma 7.1. $\mathrm{Hom}_{\mathcal{U}(\mathfrak{sl}_2(\Bbbk))}(\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk),A)\cong\Bbbk^2$. *More precisely, for any*

$$\varphi\in\mathrm{Hom}_{\mathcal{U}(\mathfrak{sl}_2(\Bbbk))}(\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk),A),$$

there exist unique $\lambda,\mu\in\Bbbk$, *such that*

$$\begin{aligned}&\varphi|_{V_5}=0,\\ &\varphi(h\otimes e-e\otimes h)=\lambda e,\quad \varphi(e\otimes f-f\otimes e)=\frac{\lambda}{2}h,\\ &\varphi(f\otimes h-h\otimes f)=\lambda f,\quad \varphi(2e\otimes f+h\otimes h+2f\otimes e)=\mu 1_A.\end{aligned}$$

In fact, the image of φ on the basis of $\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk)$ is shown in Table 1.

To compute the FGV-Poisson cohomology of A, we also need the following facts.

Lemma 7.2. *Keep the above notations. Then*

(1) $\mathrm{Ext}^2_{\mathcal{U}(A)}(\Bbbk,A)=0$;
(2) $\mathrm{Ext}^1_{\mathcal{U}(A)}(\Bbbk,A)\cong\Bbbk$;
(3) $\mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A),A)\cong\Bbbk^2$.

Proof. The proofs of (1) and (2) are just some routine calculations and are omitted here. We only prove the part (3), which needs some technical argument.

By definition, $\Omega^2(A)=A^4/I$, where I is the submodule of the $\mathcal{Q}$-module A^4 generated by

$$\{a\otimes b\otimes c\otimes 1_A-1\otimes ab\otimes c\otimes 1_A+1\otimes a\otimes bc\otimes 1_A-1\otimes a\otimes b\otimes c\}.$$

Therefore, we know that

$$\mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A),A)\cong\{f\in\mathrm{Hom}_{\mathcal{U}(A)}(A^2,A)\mid f \text{ satisfies } (*)\},$$

where $(*)$ means the equation

$$af(b,c)-f(ab,c)+f(a,bc)-f(a,b)c=0, \tag{$*$}$$

or equivalently, f is a 2-cocycle in the Hochschild complex.

Suppose that φ is a $\Bbbk$-bilinear map satisfying $(*)$ in $\mathrm{Hom}_{\mathcal{U}(A)}(A^2,A)$. Then we have $\varphi(1_A,1_A)\in Z(A)$ and hence $\varphi(1_A,1_A)=\nu 1_A$ for some $\nu\in\Bbbk$. The reason is that φ is a Lie module homomorphism, i.e.,

$$\{a,\varphi(1_A\otimes 1_A)\}=\varphi(\{a,1_A\otimes 1_A\})=0$$

holds for all $a\in A$. By applying $(*)$, we therefore obtain that for all $x\in A$, $\varphi(x\otimes 1_A)=\varphi(1_A\otimes x)=\nu x$.

On the other hand, each Lie module can be viewed as a module over $\mathfrak{sl}_2(\Bbbk)$ and

$$\mathrm{Hom}_{\mathcal{U}(A)}(A^2,A)=\mathrm{Hom}_{\mathcal{U}(\mathfrak{sl}_2(\Bbbk))}(A^2,A).$$

Now φ is determined by ν and its restriction to $\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk)$, and the latter one is uniquely given by some λ and μ as shown in Lemma 7.1. By Table 1 and the equation $(*)$, we show that $\mu=3\lambda$. Therefore, φ is determined by ν and λ as shown in Table 2.

Conversely, any $\nu,\lambda\in\Bbbk$ uniquely give to an element $\Phi_{\nu,\lambda}$ in $\mathrm{Hom}_{\mathcal{U}(A)}(A^2,A)$ which satisfies $(*)$, and hence an element in $\mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A),A)$. The proof is completed. □

Table 1 The image of φ on the basis of $\mathfrak{sl}_2(\Bbbk)\otimes\mathfrak{sl}_2(\Bbbk)$

$\varphi(-\otimes-)$	e	f	h
e	0	$\frac{\mu}{6}1_A+\frac{\lambda}{4}h$	$-\frac{\lambda}{2}e$
f	$\frac{\mu}{6}1_A-\frac{\lambda}{4}h$	0	$\frac{\lambda}{2}f$
h	$\frac{\lambda}{2}e$	$-\frac{\lambda}{2}f$	$\frac{\mu}{3}1_A$

Table 2 The image of φ on the $\Bbbk$-basis of $A\otimes A$

$\varphi(-\otimes-)$	1_A	e	f	h
1_A	$\nu 1_A$	νe	νf	νh
e	νe	0	$\frac{\lambda}{2}1_A+\frac{\lambda}{4}h$	$-\frac{\lambda}{2}e$
f	νf	$\frac{\lambda}{2}1_A-\frac{\lambda}{4}h$	0	$\frac{\lambda}{2}f$
h	νh	$\frac{\lambda}{2}e$	$-\frac{\lambda}{2}f$	$\lambda\ 1_A$

Table 3 The image of m_t on the basis of $A\otimes A$

$m_t(-,-)$	1_A	e	f	h
1_A	1_A	e	f	h
e	e	0	$\frac{1}{2}(1-t)h+\frac{1}{2}(1-ts)^2 1_A$	$-e-tse$
f	f	$-\frac{1}{2}(1-t)h+\frac{1}{2}(1-ts)^2 1_A$	0	$f+tsf$
h	h	$e-tse$	$-f-tsf$	$(1-ts)^2$

As a vector space, $\mathbb{M}_2(\Bbbk)$ is of dimension 4, and hence there are many choices of the associative multiplication on it, among which are two extreme cases. One is given by the matrix product, and the other one is trivial. An interesting observation is that these two products are essentially the only cases to make the general linear Lie algebra $\mathbb{M}_2(\Bbbk)$ a Poisson algebra.

Corollary 7.3. *Let* $(\mathbb{M}_2(\Bbbk),\circ,[-,-])$ *be a Poisson algebra, where* $(\mathbb{M}_2(\Bbbk),[-,-])$ *is the general linear Lie algebra. Then as an associative algebra,* $(\mathbb{M}_2(\Bbbk),\circ)$ *is either isomorphic to the matrix algebra, or to the trivial one.*

Proof. For simplicity, we set $P=\mathbb{M}_2(\Bbbk)$ and $m(a,b)=a\circ b$ for $a,b\in P$. The Leibniz rule implies that m is a homomorphism of Lie modules from P^2 to P. From the proof of Lemma 7.2(4) and Table 1, we know $4\mu=3\lambda^2$ since $m(a,m(b,c))=m(m(a,b),c)$. When $\lambda=0$, the associative algebra $(\mathbb{M}_2(\Bbbk),\circ)$ is the trivial associative algebra; when $\lambda\neq 0$, it is isomorphic to the matrix algebra. □

By applying the long exact sequence in Theorem 6.4 and Lemma 7.2, the Poisson cohomology groups of A of lower degrees are calculated as follows.

Proposition 7.4. *Keep the above notations. Then*

(1) $\mathrm{HP}^0(A)\cong\Bbbk$;

(2) $\mathrm{HP}^1(A)=0$;

(3) $\mathrm{HP}^2(A)\cong\Bbbk$.

More precisely, $\overline{(\Phi_{0,2},0)}$ gives a basis of $\mathrm{HP}^2(A)$. In fact, by construction as in Table 2, $\Phi_{0,2}$ is a Lie module homomorphism and satisfies the condition $(*)$, which implies that $(\Phi_{0,2},0)\in\mathrm{Hom}(A^2\oplus\wedge^2,A)$ is a Poisson 2-cocycle of A, and its corresponding cohomology class $\overline{(\Phi_{0,2},0)}$ gives an element in $\mathrm{HP}^2(A)$.

For given $s\in\Bbbk$, we may define a $\Bbbk[[t]]$-Poisson algebra structure on $A[[t]]$ by setting $l_i=0$ for all $i\geqslant 0$, and m_t to be given as in Table 3.

It is direct to show that $(A[[t]],m_t,l_t)$ is a formal Poisson deformation of A "lifting" the Poisson 2-cocycle $s(\Phi_{0,2},0)=(\Phi_{0,2s},0)$, i.e., $(m_1,l_1)=(\Phi_{0,2s},0)$. Moreover, for any Poisson 2-cocycle η, there exists some $s\in\Bbbk$ such that $\bar{\eta}=\overline{(\Phi_{0,2s},0)}$ in $\mathrm{HP}^2(A)$, and a standard argument shows the existence of a formal deformation $(A[[t]],m'_t,l'_t)$ lifting η which is equivalent to $(A[[t]],m_t,l_t)$.

Remark 7.5. It is worth mentioning that for the given Poisson 2-cocycle $(\Phi_{0,2s},0)$, we only give one formal deformation. It is not known yet whether all the formal deformation liftings of the above Poisson 2-cocycle are equivalent.

By the long exact sequence (6.1) in Theorem 6.4 and some direct calculations, we obtain $\mathrm{HP}^3(A)\neq 0$. Therefore, the above example also tells that it is still possible to have formal deformations even though "obstructions" exist.

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